International Journal of Theoretical Physics, Vol. 27, No. 2, 1988

Bell's Theorem Does Not Eliminate Fully Causal Hidden Variables

Carl H. Brans[1]

Received August 12, 1987

Bell's theorem applies only to a hybrid universe in which hidden variables determine only part of the outcomes of experiments. When applied to a fully causal hidden variable theory, in which detector settings as well as their interaction with particles during observation are determined by the variables, Bell's analysis must be modified. The result is that a fully causal hidden variable model can be produced for which a properly chosen spread of hidden variables gives precisely the same prediction as standard quantum theory.

Bell's theorem[2] is a noteworthy milestone in the continuing evolution of our understanding of the foundations of quantum theory. Bell's argument is elegant in its simplicity of logic and mathematics. It is generally believed that this result eliminates any causal hidden variable theory as a viable replacement for the intrinsic and inevitable probability spread of experimental outcomes for most subatomic experiments as predicted by the standard Copenhagen interpretation of quantum theory. Bell's argument can be summarized as follows: A hidden variable theory is one that assumes the existence of certain variables λ, which are at present "hidden" from us for some reason, but completely determine the outcome of each experiment. Furthermore, there are assumed to be certain Cauchy-type evolutionary equations that determine the future values of λ uniquely from their initial values at any arbitrary time. The details of these equations are also hidden from us, but exist in principle.[3] A hidden variable theory would then claim

[1]Physics Department, Loyola University, New Orleans, Louisiana 70118.

[2]The original, and perhaps clearest, statement is in Bell (1964). For a less technical and more verbal presentation see Mermin (1985).

[3]As an example, consider interacting charges in an electromagnetic field, satisfying classical mechanics and Maxwell's equations. Here the variables λ are the charge, mass, phase-space coordinates of all particles, plus the values of the electromagnetic fields. The coupled Lorentz force law and Maxwell equations provide the unique time evolution of the system. For a macroscopic system such as a plasma, we must employ a statistical average over λ, but only because of ignorance of fully determined details.

to be able to produce the probabilistic statistical predictions of quantum theory by assigning some appropriate statistical spread $\rho(\lambda)$ to the initial values of λ. The probabilities of quantum theory then become no more mysterious than those used in classical statistical mechanics, and in both cases would merely be due to our experimental limitations in the collection of initial data. Bell admits that such a theory might be viable for the predictions concerning the outcome of experiments obtaining the values of one localized observable for a single particle. However, his argument apparently demolishes *a priori* any such theory when applied to an experiment collecting data from distant, causally separated detection events for a pair of particles originally correlated (e.g., being in a singlet spin state), provided that the following, crucial assumption is made:

> BA: The choices of detection details of the experiment (e.g., spin direction) are made independently of each other, and of the production of the particles.

With this assumption, the argument is straightforward. The statistical spread of λ described by a density $\rho(\lambda)$ must be independent of the detector choices. If so, no linear averaging of experimental outcomes can reproduce the prediction of quantum theory, which is indeed consistent with experiment.

Bell's assumption has been variously expressed in terms of having the detector settings chosen "freely" or "randomly." The key point is that they are assumed to be independent of the event correlating the particles, i.e., their preparation in a given initial state. The main purpose of this paper is to emphasize that Bell's assumption is in fact inconsistent with a fully causal hidden variable theory in which, following classical determinism:

> FCA: All aspects of the experiment, including detector settings, are determined by initial data at some sufficiently remote time.

In other words, in a truly classical mechanical hidden variable theory, there are no "free" or "random" events, but only events whose determining variables are not known in sufficient detail.

Although, as we will see in more detail below, the physical significance of Bell's theorem rests entirely on the use of the assumption BA to the exclusion of FCA, little notice has been given to this point in publications. Others (Stapp, 1980; Clauser and Shimony, 1978; Bell, 1981; Peres, 1978; Finkelstein, 1987), have indeed pointed it out, but generally relegate discussion of the matter to a few sentences, generally claiming that FCA is too preposterous to be considered. Typical is Bell's (1981) statement that it (FCA) would be "more mind boggling than causal chains that go faster than light." Furthermore, widely read popular discussions of the subject, such as Mermin's (1985), generally make no mention at all of the tacit assumption of "reasonableness," in addition to subluminal propagation, etc., that are imposed on the hidden variables to eliminate FCA in favor of BA.

The main purpose of this paper is to reexamine this issue in more detail and hopefully without prejudice as to what structure might be more "mind boggling" than another. The aim is not so much to advocate any particular hidden variable theory, but rather to point out that it is quite simply false to claim that fully causal hidden variable theories, modeled after classical mechanical causality, are excluded by Bell's theorem and related experimentation.

Let us begin by reviewing the physical and mathematical setting of Bell's theorem. The idealized experimental situation refers to a pair of spin-1/2 particles prepared in a singlet state in a region R and then observed by spin detectors D_1 and D_2 in regions R_1 and R_2. Regions R_1 and R_2 are distantly separated so that light signals cannot connect either detection with the other. Furthermore, and this is the crucial point, the emission of the pair at event E at region R cannot be causally connected to the selection of the directions **a** and **b** for D_1 and D_2 at events S_1 and S_2, respectively. This is schematically represented in the space-time diagram of Figure 1.

Implicit in the discussion of a single outcome of this experiment is the assumption that the preparation of the particles at the emission event corresponds to a particular value λ_E of the hidden variables. Some deterministic law then uniquely determines their value λ_D at the later detection time. For simplicity, it can be assumed that $\lambda_D = \lambda_E$, although this is not

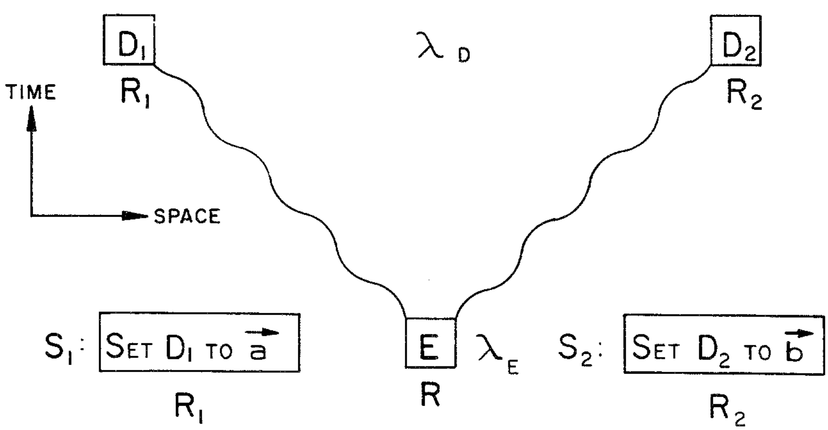

Fig. 1

relevant to the argument. The statistical results obtained in many performances of the experiment are then to be ascribed to some distribution $\rho(\lambda_E)$ corresponding to the preparation of the particles in the singlet state many times. The point at issue in this paper is Bell's *apparently* natural assumption that causality requires that ρ be a function of λ_E only, and explicitly independent of the choices of **a** and **b** for the detector orientations, since these choices are made at events S_1 and S_2 that are causally independent of the preparation at E. In other words, while a hidden variable theory might assert that there is some spread in initial values for λ_E in the several preparations of the particles, this spread cannot depend on the choice of **a** and **b**, since in some frame, the preparation and choices are simultaneous.

However, implicit in this is the additional assumption that the hidden variables λ do not themselves determine the settings **a** and **b** for D_1 and D_2. Actually, in a fully causal hidden variable theory (FCA), the detectors, and in particular their orientations **a** and **b**, are themselves part of the complete experiment and thus subject to the deterministic evolutionary laws governing λ and hence the full outcome of each experimental repetition. Consider then Figure 2 as a completion of Figure 1. The question left unanswered by a use of BA to the exclusion of FCA is: Why stop the backward causal analysis at the time of S_1 , E, S_2 as in Figure 1, rather than going to Figure 2?

From now on the significance of λ will be expanded from variables describing hidden properties of the particle pair alone to variables describing

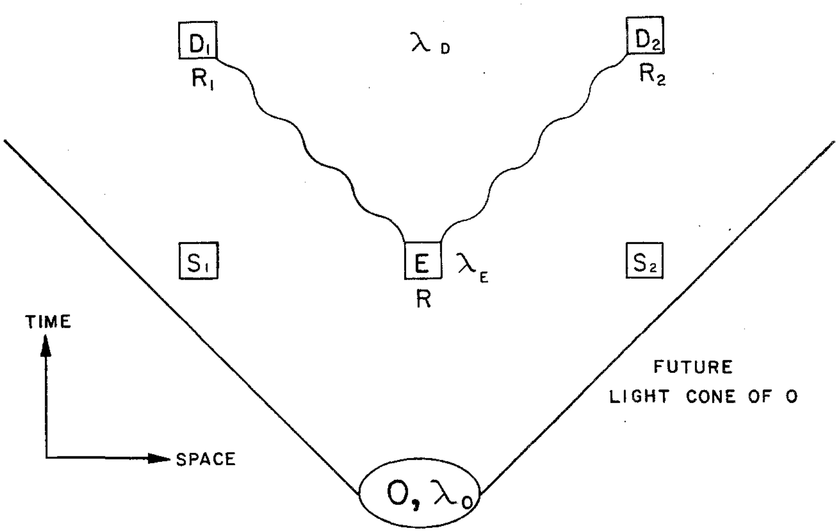

Fig. 2

the selection of settings **a** and **b** for D_1 and D_2 as well. The region O has a future light cone encompassing not only the detection events, but also the events S_1 and S_2 corresponding to the selection of **a** and **b** for D_1 and D_2. The region O can be the region consisting of the laboratory at the time of the preparation of all of the equipment.

Consider now a restatement of Bell's equation governing the hidden variable prediction for the accumulated average of measurements of the spins at D_1 and D_2,

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda_E \, \rho(\lambda_E) A(\mathbf{a}, \lambda_D) B(\mathbf{b}, \lambda_D) \tag{1}$$

Here $A(\mathbf{a}, \lambda_D)$ and $B(\mathbf{b}, \lambda_D)$ represent the determined outcome of the spin measurements corresponding to the value of λ_D of the hidden variables at detection time. These are assumed to be fully determined by the values λ_E at emission time and the integral over λ_E weighted by ρ describes the spread of individual results corresponding to the various repetitions of the experiment. When considered from the viewpoint of Figure 2, however, equation (1) must be replaced by one of the form

$$P(\mathbf{a}, \mathbf{b}) = \frac{[\int d\lambda_0 \, \rho(\lambda_0) A(\mathbf{a}, \lambda_D) B(\mathbf{b}, \lambda_D) \; \delta(\lambda_D \in (\mathbf{a}, \mathbf{b}))]}{[\int d\lambda_0 \, \rho(\lambda_0) \; \delta(\lambda_D \in (\mathbf{a}, \mathbf{b}))]} \tag{2}$$

Here the λ integration is to be understood in terms of distributions and $\delta(\lambda_D \in (\mathbf{a}, \mathbf{b}))$ is the delta fuction for λ_D having values resulting in the setting of **a** and **b** for D_1 and D_2. Expressed verbally, equation (2) says that the average values of the measurement of D_1 times that of D_2 for the subset of outcomes corresponding to the settings **a** and **b** for D_1 and D_2 is the integral of the product of those outcomes that result in **a** and **b**, divided by the probability that **a** and **b** occur. Bell's formula is based on an average assocated with an *absolute* probability, whereas, since it only pertains to the subclass of experiments corresponding to the outcome **a** and **b** for D_1 and D_2, it should be replaced by an average based on *conditional* probability, equation (2). Note that (2) can also be written

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda_0 \, \bar{\rho}(\lambda_0) A(\mathbf{a}, \lambda_D) B(\mathbf{b}, \lambda_D) \tag{3}$$

where $\bar{\rho}$ *now depends on a and b,*

$$\bar{\rho}(\lambda_0) = \rho(\lambda_0) \; \delta(\lambda_D \in (\mathbf{a}, \mathbf{b})) \Big/ \int d\lambda_0 \, \rho(\lambda_0) \; \delta(\lambda_D \in (\mathbf{a}, \mathbf{b})) \tag{4}$$

Of course, the deterministic evolution assumption for λ implies that λ_0 is a function of λ_D.

To make explicit the involvement of the choice of directions **a** and **b** as part of the fully determined experimental outcome governed by λ and their evolutionary equations, replace λ by $(\lambda', \mathscr{A}, \mathscr{B})$, where $\mathscr{A}$ and $\mathscr{B}$ are unit vectors describing the parts of the hidden variables determining the settings of D_1 and D_2. Thus, (3) becomes

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda_0'\, d^2\mathscr{A}\, d^2\mathscr{B}\, \bar{\rho}(\lambda_0', \mathscr{A}, \mathscr{B}) A(\mathbf{a}, \lambda_D) B(b, \lambda_D) \tag{5}$$

All that now remains to be done is to show that the quantum mechanical result,

$$P(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} \tag{6}$$

can be reproduced by appropriately chosen $\bar{\rho}$ in (5). For the example considered, let $|\phi_\pm^{\mathbf{c}}\rangle$ be the one-particle quantum eigenstates of a spin detector in the direction **c**. If **z** is any arbitrary but fixed direction and $\alpha, \beta = \pm 1$ are the spin eigenvalues (normalized to $\hbar = 2$), then set

$$\bar{\rho}(\lambda', \mathscr{A}, \mathscr{B}) = c_{ij}^* c_{kl} \langle \phi_i^{\mathbf{z}} \mid \phi_\alpha^{\mathscr{A}} \rangle \langle \phi_\alpha^{\mathscr{A}} \mid \phi_k^{\mathbf{z}} \rangle \cdot \langle \phi_j^{\mathbf{z}} \mid \phi_\beta^{\mathscr{B}} \rangle \langle \phi_\beta^{\mathscr{B}} \mid \phi_l^{\mathbf{z}} \rangle\, \delta(\mathscr{A} - \mathbf{a})\, \delta(\mathscr{B} - \mathbf{b}) \tag{7}$$

(no sum)

where

$$c_{++} = c_{--} = 0, \qquad c_{+-} = -c_{-+} = 1/\sqrt{2} \tag{8}$$

and we are explicitly replacing λ' by $(i, j, k, l, \alpha, \beta)$, all $= \pm 1$. Furthermore,

$$\int d\lambda' \to \sum_{i,j,k,l,\alpha,\beta}, \qquad A(\mathbf{a}, \lambda) = \alpha; \qquad B(\mathbf{b}, \lambda) = \beta \tag{9}$$

Here it is assumed that $\lambda_D = \lambda_E = \lambda_0$ for simplicity. It is then straightforward algebra to show that

$$\int d\lambda\, \bar{\rho}(\lambda) = 1; \qquad \bar{\rho}(\lambda) \geq 0 \tag{10}$$

and that (5) gives exactly the quantum prediction (6). This may be easily seen by going back to the original ρ setting

$$\rho(\lambda', \mathscr{A}, \mathscr{B}) = |\langle \psi | (|\phi_\alpha^{\mathscr{A}}\rangle \otimes |\phi_\beta^{\mathscr{B}}\rangle)|^2 / (4\pi)^2 \tag{11}$$

where

$$|\psi\rangle = \sum_{ij} c_{ij} |\phi_i^{\mathbf{z}}\rangle \otimes |\phi_j^{\mathbf{z}}\rangle \tag{12}$$

and

$$\int d^2\mathscr{A}\, d^2\mathscr{B} = (4\pi)^2 \tag{13}$$

Let us now consider the physical implications of this formalism in more detail. The part of the argument that seems to cause most conceptual problems involves the apparent lack of independence of the detector settings from each other and from the particle emission. Thus, in (7) the density for the hidden variables indeed depends not only on the state of the particle pair through c_{ij}, but also on the detector settings **a** and **b**. It should be recalled that the entry of the detector setting values into λ was through the fact that we are computing a *conditional* rather than *absolute* average. We thus use conditional probability, selecting only those outcomes, i.e., deterministic hidden variable values, that give rise to detector settings **a** and **b**, respectively. Nevertheless, there seems to be a very deep prejudice that while what goes on in the emission and propagation of the particle pair may be deterministic, the settings for D_1 and D_2 are not! We can only repeat again that true "free" or "random" behavior for the choice of detector settings is inconsistent with a fully causal set of hidden variables. How can we have part of the universe determined by λ and another part not? However, given FCA, the choice of the settings at S_1 and S_2 cannot be regarded as uncorrelated in the sense of being unpredictable from sufficiently earlier data in a complete theory. Opponents, however, may still object that such correlation is "unreasonable" and inconsistent with our experience of "free" and "independent" choices.

Look at these points in more detail. First, a literal use of "free" necessarily takes us out of the realm of what is generally regarded as physical science and into metaphysics. However, some operational sense can be made of "free" and "independent" in terms of statistical measures. This still would cause no difficulty for FCA. The arguments above are entirely consistent with the outcome that $\mathbf{a}_i$ and $\mathbf{b}_i$ are "random" functions of i, the experiment number, $i = 1, 2, \ldots$, and that there is no statistical corelation between $\mathbf{a}_i$ and $\mathbf{b}_i$. All that the FCA formalism uses is that, for those i for which $\mathbf{a}_i = \mathbf{a}$ and $\mathbf{b}_i = \mathbf{b}$, the distribution is given by (7).

It is sometimes said that quantum theory saves free will. In the context of this paper, this might be reversed, so that free will saves quantum theory, at least in the sense of eliminating hidden variable alternatives. In other words, if there are any truly "free" events in the experiment, then there can be no classical determinism and hence no classical hidden variables. Conversely, given FCA, there are no truly "free" or "random" events, although certain sets of variable values may be uncorrelated in any contemporary statistical sense. Thus, an FCA type of hidden variable theory can reproduce exactly the predictions of quantum theory, yet still preserve the apparent randomness of certain choices.

ACKNOWLEDGMENT

I am grateful to A. R. Marlow for helpful discussions.

REFERENCES

Bell, J. S. (1964). *Physics*, **1**, 195.
Bell, J. (1981). *Journal de Physique, Colloque C2* (Suppl. 3), **42**, 57.
Clauser, J. F., and Shimony, A. (1978). *Reports on Progress in Physics*, **41**, 1881.
Finkelstein, D. (1987). In *Encyclopedia Britannica Yearbook of Science*, p. 186.
Mermin, N. (1985). *Physics Today* **38**, 38.
Peres, A. (1978). *American Journal of Physics*, **46**, 745.
Stapp, H. P. (1980). *Foundations of Physics*, **10**, 767.